An Error Analysis Algorithm for Approximate Solution of Linear Fredholm-Stieltjes Integral Equations with Generalized Trapezium Method

Isa Muslu, Ph.D.
The Department of Applied Mathematics and Computer
International Ataturk-Alatoo University
Bishkek, Kyrgyzstan
isa.muslu@iaau.edu.kg

Moussa Mahamat Boukar, Ph.D.
The Department of Computer Engineering
Nigerian Turkish Nile University
Abuja, Nigeria
musa.muhammed@nileuniversity.edu.ng

Shahriyar Shamiluulu
Department of Computer Sciences
Suleiman Demirel University
Almaty, Kazakhstan
Shahriar.shamiluulu@sdu.edu.kz

***Abstract*—Integral equations and their solutions are very important for various areas like physics, engineering, biology and other. Fredholm-Stieltjes integral equations are some of the integral equations. Sometimes it is possible to find exact solutions for some of the integral equations.The main purpose of this paper is to propose an error analysis algorithm for approximate solution of linear Fredholm-Stieltjes integral equations of second kind with Generalized Trapezium Method. Firstly, the theory of error analysis is given. Then the implementation of algorithm is done with Maple software and examples are given with graphics.**

***Index Terms*—Integral Equations; Stieltjes Integral Equations; Approximation Methods; Generalized Trapezium Method; Error Analysis for Integral Equations**

I. INTRODUCTION

Integral equations are used in many various areas [1-5] like electromagnetic theory [6], thermoelectricity [7], biology, and mechanics [8]. Various analytic [9-11] and numeric [12-14] solutions have been developed in the history. Stieltjes integral equations are one of the big class in the integral equations and classified as Fredholm and Volterra. There are a few works about Stieltjes integral equations in the literature [15-16]. Fredholm-Stieltjes integral equations are one of the most used equations and it is possible to find some works about its solutions [17-19]. In this work the error analysis is done for the linear Fredholm-Stieltjes integral equations of second kind solved by the trapezium approximate solution method [20]. In this method, the equation is transformed to an algebraic equations system and this system is solved.

II. PROBLEM STATEMENT

Consider

$$\begin{cases} u(x) = \frac{\lambda}{2}\sum_{i=1}^{n}\left[K(x,x_i)u(x_i) + K(x,x_{i-1})u(x_{i-1})\right]\left[\varphi(x_i) - \varphi(x_{i-1})\right] - \\ \quad -\frac{\lambda}{2}\sum_{i=1}^{n}\left[K(x,x_i)u(x_i) + K(x,x_{i-1})u(x_{i-1})\right]\left[\psi(x_i) - \psi(x_{i-1})\right] + \\ \quad f(x) + E(T) \end{cases} \tag{1}$$

as a solution of the equation

$$u(x) = f(x) + \int_a^b K(x,s)u(s)\,dg(s);\ x \in [a,b]\ s \in [a,b] \tag{2}$$

If (2) is written in the form of

$$u(x) = \frac{\lambda}{2}\sum_{i=1}^{n}\begin{bmatrix} K(x,x_i)u(x_i) + K(x,x_{i-1})u(x_{i-1}) \\ \left(\varphi(x_i) - \varphi(x_{i-1})\right) - \left(\psi(x_i) - \psi(x_{i-1})\right) \end{bmatrix} + f(x) + \lambda E(h) \tag{3}$$

, then the error formula can be written as

$$E(h) = \frac{M_0}{12}\sum_{i=1}^{n}\left(\omega_\varphi(h)\right)^2\left(\varphi(x_i) - \varphi(x_{i-1})\right) + \frac{M_1}{12}\sum_{i=1}^{n}\left(\omega_\psi(h)\right)^2\left(\psi(x_i) - \psi(x_{i-1})\right) \tag{4}$$

where

$$M_0(x) = \sup_{s\in[a,b]}\left|\left(K(x,s)u(x)\right)''_{\varphi(x)}\right| = \sup_{s\in[a,b]}\left|K''_{\varphi(x)}(x,s)u(x) + 2K'_{\varphi(x)}(x,s)u'_{\varphi(x)}(x) + K(x,s)u''_{\varphi(x)}(x)\right| \tag{5}$$

and

$$M_1(x)=\sup_{s\in[a,b]}\left|\left(K(x,s)u(x)\right)''_{\psi(x)}\right|= \tag{6}$$
$$=\sup_{s\in[a,b]}\left|K''_{\psi(x)}(x,s)u(x)+2K'_{\psi(x)}(x,s)u'_{\psi(x)}(x)+K(x,s)u''_{\psi(x)}(x)\right|$$

To find expanded form of (5) the norm of the equation

$$u(x)=\lambda\int_a^b K(x,s)u(s)dg(s)+f(x),\quad g(s)=\varphi(s)-\psi(s) \tag{7}$$

is taken

$$\|u(x)\|_c=\left\|\lambda\int_a^b K(x,s)u(s)dg(s)+f(x)\right\|$$

$$\|u(x)\|_c\le|\lambda|\sup_{s\in[a,b]}|K(x,s)|\cdot\|u(x)\|_c\cdot\left|\int_a^b dg(s)\right|\cdot\|f(x)\|_c$$

and is obtained

$$\|u(x)\|_c\left(1-\underbrace{|\lambda|\sup_{s\in[a,b]}|K(x,s)|\cdot\|u(x)\|_c\left(g(b)-g(a)\right)}_{\alpha}\right)\le\|f(x)\|_c \tag{8}$$

Here if $\alpha=|\lambda|\sup_{s\in[a,b]}|K(x,s)|\cdot\|u(x)\|_c\left(g(b)-g(a)\right)<1$ (9)

is taken, then the expression

$$\|u(x)\|_c\le\frac{\|f(x)\|_c}{1-\alpha} \tag{10}$$

is obtained.

If the first and second derivatives of (2) are taken and computed their norms, then the expressions

$$\left\|u'_{\varphi(x)}(x)\right\|_c\le|\lambda|\sup_{s\in[a,b]}\left|K'_{\varphi(x)}(x,s)\right|\cdot\|u(x)\|_c\cdot\left|g(b)-g(a)\right|\cdot\left\|f'_{\varphi(x)}(x)\right\|_c \tag{11}$$

$$\left\|u''_{\varphi(x)}(x)\right\|_c\le|\lambda|\sup_{s\in[a,b]}\left|K''_{\varphi(x)}(x,s)\right|\cdot\|u(x)\|_c\cdot\left|g(b)-g(a)\right|\cdot\left\|f''_{\varphi(x)}(x)\right\|_c \tag{12}$$

$$\left\|u'_{\psi(x)}(x)\right\|_c\le|\lambda|\sup_{s\in[a,b]}\left|K'_{\psi(x)}(x,s)\right|\cdot\|u(x)\|_c\cdot\left|g(b)-g(a)\right|\cdot\left\|f'_{\psi(x)}(x)\right\|_c \tag{13}$$

$$\left\|u''_{\psi(x)}(x)\right\|_c\le|\lambda|\sup_{s\in[a,b]}\left|K''_{\psi(x)}(x,s)\right|\cdot\|u(x)\|_c\cdot\left|g(b)-g(a)\right|\cdot\left\|f''_{\psi(x)}(x)\right\|_c \tag{14}$$

is received, respectively.

After substituting (11) and (12) into the (5) and some algebraic operations the expression

$$M_0(x)=\sup_{s\in[a,b]}\left|\left(K(x,s)u(x)\right)''_{\varphi(x)}\right|=$$
$$=\sup_{s\in[a,b]}\left|K''_{\varphi(x)}(x,s)u(x)+2K'_{\varphi(x)}(x,s)u'_{\varphi(x)}(x)+K(x,s)u''_{\varphi(x)}(x)\right|$$
$$\le\sup_{s\in[a,b]}\left|K''_{\varphi(x)}(x,s)\right|\|u(x)\|_c+2\cdot\sup_{s\in[a,b]}\left|K'_{\varphi(x)}(x,s)\right|\left\|u'_{\varphi(x)}(x)\right\|_c$$
$$+\sup_{s\in[a,b]}|K(x,s)|\left\|u''_{\varphi(x)}(x)\right\|$$
$$=\sup_{s\in[a,b]}\left|K''_{\varphi(x)}(x,s)\right|\frac{\|f(x)\|_c}{1-\alpha}+2\cdot\sup_{s\in[a,b]}\left|K'_{\varphi(x)}(x,s)\right|$$
$$\left(|\lambda|\cdot\sup_{s\in[a,b]}\left|K'_{\varphi(x)}(x,s)\right|\|u(x)\|_c\left|g(b)-g(a)\right|+\left\|f'_{\varphi(x)}\right\|_c\right)$$
$$+\sup_{s\in[a,b]}|K(x,s)|\left\|u''_{\varphi(x)}(x)\right\|+\sup_{s\in[a,b]}|K(x,s)|$$
$$\left(|\lambda|\cdot\sup_{s\in[a,b]}\left|K''_{\varphi(x)}(x,s)\right|\|u(x)\|_c\left|g(b)-g(a)\right|+\left\|f''_{\varphi(x)}\right\|_c\right)$$
$$=\frac{\|f(x)\|_c}{1-\alpha}\left[\begin{array}{l}\sup_{s\in[a,b]}\left|K''_{\varphi(x)}(x,s)\right|+2\cdot|\lambda|\cdot\left(\sup_{s\in[a,b]}\left|K'_{\varphi(x)}(x,s)\right|\right)^2\cdot\left|g(b)-g(a)\right|+\\+|\lambda|\cdot\sup_{s\in[a,b]}|K(x,s)|\cdot\sup_{s\in[a,b]}\left|K''_{\varphi(x)}(x,s)\right|\left|g(b)-g(a)\right|\end{array}\right] \tag{15}$$
$$+2\cdot\sup_{s\in[a,b]}\left|K'_{\varphi(x)}(x,s)\right|\cdot\left\|f'_{\varphi(x)}\right\|_c+\sup_{s\in[a,b]}|K(x,s)|\cdot\left\|f''_{\varphi(x)}\right\|_c$$

is got.

If the same operations is done to find $M_1(x)$, that is, (13) and (14) are substituted into (6) and is done some algebraic operations, then the expression

$$M_1(x)=\sup_{s\in[a,b]}\left|\left(K(x,s)u(x)\right)''_{\psi(x)}\right|=$$
$$=\sup_{s\in[a,b]}\left|K''_{\psi(x)}(x,s)u(x)+2K'_{\psi(x)}(x,s)u'_{\psi(x)}(x)+K(x,s)u''_{\psi(x)}(x)\right|$$
$$\le\sup_{s\in[a,b]}\left|K''_{\psi(x)}(x,s)\right|\|u(x)\|_c+2\cdot\sup_{s\in[a,b]}\left|K'_{\psi(x)}(x,s)\right|\left\|u'_{\psi(x)}(x)\right\|_c$$
$$+\sup_{s\in[a,b]}|K(x,s)|\left\|u''_{\psi(x)}(x)\right\|$$

$$= \sup_{s\in[a,b]} \left|K''_{\psi(x)}(x,s)\right| \frac{\|f(x)\|_c}{1-\alpha} + 2\cdot\sup_{s\in[a,b]}\left|K'_{\psi(x)}(x,s)\right|$$

$$\left(|\lambda|\cdot\sup_{s\in[a,b]}\left|K'_{\psi(x)}(x,s)\right|\|u(x)\|_c\,|g(b)-g(a)|+\left\|f'_{\psi(x)}\right\|_c\right)$$

$$+\sup_{s\in[a,b]}|K(x,s)|\left\|u''_{\psi(x)}(x)\right\|+\sup_{s\in[a,b]}|K(x,s)|$$

$$\left(|\lambda|\cdot\sup_{s\in[a,b]}\left|K''_{\psi(x)}(x,s)\right|\|u(x)\|_c\,|g(b)-g(a)|+\left\|f''_{\psi(x)}\right\|_c\right)$$

$$=\frac{\|f(x)\|_c}{1-\alpha}\left[\begin{array}{l}\sup\limits_{s\in[a,b]}\left|K''_{\psi(x)}(x,s)\right|+2\cdot|\lambda|\cdot\left(\sup\limits_{s\in[a,b]}\left|K'_{\psi(x)}(x,s)\right|\right)^2\cdot|g(b)-g(a)|+\\+|\lambda|\cdot\sup\limits_{s\in[a,b]}|K(x,s)|\cdot\sup\limits_{s\in[a,b]}\left|K''_{\psi(x)}(x,s)\right||g(b)-g(a)|\end{array}\right] \tag{16}$$

$$+2\cdot\sup_{s\in[a,b]}\left|K'_{\psi(x)}(x,s)\right|\cdot\left\|f'_{\psi(x)}\right\|_c+\sup_{s\in[a,b]}|K(x,s)|\cdot\left\|f''_{\psi(x)}\right\|_c$$

is obtained. The quantity E(h) can be computed by replacing (15) and (16) in the (4). Because of the equation is satisfied at every quadrature point, $x = x_i$ is taken and replaced in the (3) and got the equation

$$u(x_i)=\frac{\lambda}{2}\sum_{j=1}^{n}\left[K(x_i,x_j)u(x_j)+K(x_i,x_{j-1})u(x_{j-1})\right]\cdot\left[\left(\varphi(x_j)-\varphi(x_{j-1})\right)-\left(\psi(x_j)-\psi(x_{j-1})\right)\right]+f(x_i)+E(h) \tag{17}$$

is received. If

$$u_i=\frac{\lambda}{2}\sum_{j=1}^{n}\left[K(x_i,x_j)u_j+K(x_i,x_{j-1})u_{j-1}\right]\cdot\left[\left(\varphi(x_j)-\varphi(x_{j-1})\right)-\left(\psi(x_j)-\psi(x_{j-1})\right)\right]+f(x_i) \tag{18}$$

is considered in (17), then the error for approximate solution is computed as

$$u(x_i)-u_i=v_i \tag{19}$$

When $u(x_i)$ and u_i are written in (19) then the general equation

$$v_i=\frac{\lambda}{2}\sum_{j=1}^{n}\left[K(x_i,x_j)\left(u(x_j)-u_j\right)+K(x_i,x_{j-1})\left(u(x_{j-1})-u_{j-1}\right)\right]\cdot\left[\left(\varphi(x_j)-\varphi(x_{j-1})\right)-\left(\psi(x_j)-\psi(x_{j-1})\right)\right]+\lambda\cdot E(h) \tag{20}$$

is obtained. Here

$$\begin{aligned}v_i&=\frac{\lambda}{2}\sum_{j=1}^{n}\left[K(x_i,x_j)u(x_j)+K(x_i,x_{j-1})u(x_{j-1})\right]\cdot\\&\cdot\left[\left(\varphi(x_j)-\varphi(x_{j-1})\right)-\left(\psi(x_j)-\psi(x_{j-1})\right)\right]+\\&+f(x_i)+\lambda E(h)-\frac{\lambda}{2}\sum_{j=1}^{n}\left[K(x_i,x_j)u_j+K(x_i,x_{j-1})u_{j-1}\right]\cdot\\&\cdot\left[\left(\varphi(x_j)-\varphi(x_{j-1})\right)-\left(\psi(x_j)-\psi(x_{j-1})\right)\right]-f(x_i)\end{aligned} \tag{21}$$

and $u(x_j)-u_j=v_i, u(x_{j-1})-u_{j-1}=v_{j-1}$

Substituting $u(x_j)-u_j=v_i$, $u(x_{j-1})-u_{j-1}=v_{j-1}$ into (21) gives

$$v_i=\frac{\lambda}{2}\sum_{j=1}^{n}\left[K(x_i,x_j)v_j+K(x_i,x_{j-1})v_{j-1}\right]\cdot\left[\left(\varphi(x_j)-\varphi(x_{j-1})\right)-\left(\psi(x_j)-\psi(x_{j-1})\right)\right]+\lambda E(h) \tag{22}$$

Replacing integers from 0 to n in the (22) results the algebraic equations system

$$\begin{cases}v_0=\frac{\lambda}{2}\sum\limits_{j=1}^{n}\begin{bmatrix}K(x_0,x_j)v_j+K(x_0,x_{j-1})v_{j-1}\\\left[\left(\varphi(x_j)-\varphi(x_{j-1})\right)-\left(\psi(x_j)-\psi(x_{j-1})\right)\right]\end{bmatrix}+\lambda E(h)\\v_1=\frac{\lambda}{2}\sum\limits_{j=1}^{n}\begin{bmatrix}K(x_1,x_j)v_j+K(x_1,x_{j-1})v_{j-1}\\\left[\left(\varphi(x_j)-\varphi(x_{j-1})\right)-\left(\psi(x_j)-\psi(x_{j-1})\right)\right]\end{bmatrix}+\lambda E(h)\\\cdots\\\cdots\\v_{n-1}=\frac{\lambda}{2}\sum\limits_{j=1}^{n}\begin{bmatrix}K(x_{n-1},x_j)v_j+K(x_{n-1},x_{j-1})v_{j-1}\\\left[\left(\varphi(x_j)-\varphi(x_{j-1})\right)-\left(\psi(x_j)-\psi(x_{j-1})\right)\right]\end{bmatrix}+\lambda E(h)\\v_n=\frac{\lambda}{2}\sum\limits_{j=1}^{n}\begin{bmatrix}K(x_n,x_j)v_j+K(x_n,x_{j-1})v_{j-1}\\\left[\left(\varphi(x_j)-\varphi(x_{j-1})\right)-\left(\psi(x_j)-\psi(x_{j-1})\right)\right]\end{bmatrix}+\lambda E(h)\end{cases} \tag{23}$$

Reorganizing (23), the expression

$$\begin{aligned}&\left(1-\frac{\lambda}{2}K(x_0,x_0)\left[\left(\varphi(x_1)-\varphi(x_0)\right)-\left(\psi(x_1)-\psi(x_0)\right)\right]\right)v_0-\\&-\frac{\lambda}{2}\sum_{j=1}^{n-1}K(x_0,x_j)\left[\left(\varphi(x_{j+1})-\varphi(x_{j-1})\right)-\left(\psi(x_{j+1})-\psi(x_{j-1})\right)\right]v_j\\&-\frac{\lambda}{2}K(x_0,x_n)\left[\left(\varphi(x_n)-\varphi(x_{n-1})\right)-\left(\psi(x_n)-\psi(x_{n-1})\right)\right]v_n=\lambda E(h)\\&-\frac{\lambda}{2}K(x_1,x_0)\left[\left(\varphi(x_1)-\varphi(x_0)\right)-\left(\psi(x_1)-\psi(x_0)\right)\right]v_0+\\&+\left(1-\frac{\lambda}{2}\sum_{j=1}^{n-1}K(x_1,x_j)\left[\left(\varphi(x_{j+1})-\varphi(x_{j-1})\right)-\left(\psi(x_{j+1})-\psi(x_{j-1})\right)\right]\right)v_j\\&-\frac{\lambda}{2}K(x_1,x_n)\left[\left(\varphi(x_n)-\varphi(x_{n-1})\right)-\left(\psi(x_n)-\psi(x_{n-1})\right)\right]v_n=\lambda E(h)\\&\cdots\cdots\\&-\frac{\lambda}{2}K(x_n,x_0)\left[\left(\varphi(x_1)-\varphi(x_0)\right)-\left(\psi(x_1)-\psi(x_0)\right)\right]v_0-\\&-\frac{\lambda}{2}\sum_{j=1}^{n-1}K(x_n,x_j)\left[\left(\varphi(x_{j+1})-\varphi(x_{j-1})\right)-\left(\psi(x_{j+1})-\psi(x_{j-1})\right)\right]v_j\\&\left(1-\frac{\lambda}{2}K(x_n,x_n)\left[\left(\varphi(x_n)-\varphi(x_{n-1})\right)-\left(\psi(x_n)-\psi(x_{n-1})\right)\right]\right)v_n=\lambda E(h)\end{aligned} \tag{24}$$

is received. The algebraic equation (24) can be written as matrix equation

$$\underbrace{\begin{pmatrix} 1-A_{00} & -A_{01} & \cdots & -A_{0(n-1)} & -A_{0n} \\ -A_{10} & 1-A_{11} & \cdots & -A_{1(n-1)} & -A_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_{(n-1)0} & -A_{(n-1)1} & \cdots & 1-A_{(n-1)(n-1)} & -A_{(n-1)n} \\ -A_{n0} & -A_{n1} & \cdots & -A_{n(n-1)} & -1-A_{nn} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} v_0 \\ v_0 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix}}_{V} = \lambda \underbrace{\begin{pmatrix} E(h) \\ E(h) \\ \vdots \\ E(h) \\ E(h) \end{pmatrix}}_{E} \tag{25}$$

by taking

$$\begin{aligned}
A_{00} &= \frac{\lambda}{2} K\left(x_0, x_0\right)\left[\left(\varphi\left(x_1\right)-\varphi\left(x_0\right)\right)-\left(\psi\left(x_1\right)-\psi\left(x_0\right)\right)\right], \\
A_{0j} &= \frac{\lambda}{2}\sum_{j=1}^{n-1} K\left(x_0, x_j\right)\left[\left(\varphi\left(x_{j+1}\right)-\varphi\left(x_{j-1}\right)\right)-\left(\psi\left(x_{j+1}\right)-\psi\left(x_{j-1}\right)\right)\right] \\
A_{0n} &= \frac{\lambda}{2} K\left(x_0, x_n\right)\left[\left(\varphi\left(x_n\right)-\varphi\left(x_{n-1}\right)\right)-\left(\psi\left(x_n\right)-\psi\left(x_{n-1}\right)\right)\right] \\
A_{10} &= \frac{\lambda}{2} K\left(x_1, x_0\right)\left[\left(\varphi\left(x_1\right)-\varphi\left(x_0\right)\right)-\left(\psi\left(x_1\right)-\psi\left(x_0\right)\right)\right] \\
A_{1j} &= \frac{\lambda}{2}\sum_{j=1}^{n-1} K\left(x_1, x_j\right)\left[\left(\varphi\left(x_{j+1}\right)-\varphi\left(x_{j-1}\right)\right)-\left(\psi\left(x_{j+1}\right)-\psi\left(x_{j-1}\right)\right)\right] \\
A_{1n} &= \frac{\lambda}{2} K\left(x_1, x_n\right)\left[\left(\varphi\left(x_n\right)-\varphi\left(x_{n-1}\right)\right)-\left(\psi\left(x_n\right)-\psi\left(x_{n-1}\right)\right)\right] \qquad (26) \\
&\cdots \\
&\cdots \\
A_{n0} &= \frac{\lambda}{2} K\left(x_n, x_0\right)\left[\left(\varphi\left(x_1\right)-\varphi\left(x_0\right)\right)-\left(\psi\left(x_1\right)-\psi\left(x_0\right)\right)\right] \\
A_{nj} &= \frac{\lambda}{2}\sum_{j=1}^{n-1} K\left(x_n, x_j\right)\left[\left(\varphi\left(x_{j+1}\right)-\varphi\left(x_{j-1}\right)\right)-\left(\psi\left(x_{j+1}\right)-\psi\left(x_{j-1}\right)\right)\right] \\
A_{nn} &= \frac{\lambda}{2} K\left(x_n, x_n\right)\left[\left(\varphi\left(x_n\right)-\varphi\left(x_{n-1}\right)\right)-\left(\psi\left(x_n\right)-\psi\left(x_{n-1}\right)\right)\right]
\end{aligned}$$

It is possible to write this matrix equation briefly as **A.V=E**
This equations system is solved for V values and substituted into the expression

$$\begin{aligned}
v(x) &= \frac{\lambda}{2} K\left(x, x_0\right) v_0\left[\left(\varphi\left(x_1\right)-\varphi\left(x_0\right)\right)-\left(\psi\left(x_1\right)-\psi\left(x_0\right)\right)\right]+ \\
&+\frac{\lambda}{2}\sum_{i=1}^{n} K\left(x, x_i\right) v_i\left[\left(\varphi\left(x_{i+1}\right)-\varphi\left(x_{i-1}\right)\right)-\left(\psi\left(x_{i+1}\right)-\psi\left(x_{i-1}\right)\right)\right] \qquad (27) \\
&+\frac{\lambda}{2} K\left(x, x_n\right) v_n\left[\left(\varphi\left(x_n\right)-\varphi\left(x_{n-1}\right)\right)-\left(\psi\left(x_n\right)-\psi\left(x_{n-1}\right)\right)\right]
\end{aligned}$$

and is arrived **v(x)** error function.
To determine an error bound for computation (25) can be used. Writing the equation (25) as

$$(I-\lambda A) V=E(h) \tag{28}$$

and leaving alone V on the left hand side gives

$$V=(I-\lambda A)^{-1} E(h) \tag{29}$$

Taking the norm and reorganizing (29) yields

$$\|V\|=\left\|(I-\lambda A)^{-1} E(h)\right\| \Rightarrow\|V\| \leq\left\|(I-\lambda A)^{-1}\right\|\|E(h)\| \tag{30}$$

and

$$\|V\| \leq \frac{\|E(h)\|}{1-|\lambda|\|A\|} \tag{31}$$

Here the condition $\left(|\lambda|\|A\|<1\right)$ must be satisfied. Thus, an error bound is found for the approximated solution of the Fredholm-Stieltjes integral equations of the second kind.

III. COMPUTATION ALGORITHM

The computation algorithm for the error analysis is as follows:

```
Omega := proc(φ, a, b, n := 100, coef_of_h := 1 )
    local x, y, h := (b − a)/n, step := h/n, sup := 0;
    for y from a by step to b − coef_of_h·h + step do
        x := y + coef_of_h· h − step;
        if |φ(x) − φ(y)| > sup then
            sup := |φ(x) − φ(y)|;
        end if;
```

Fig. 1.Algorithm for omega (h)

Error_General := proc (K, varphi, psi, f, a, b, lambda := 1, n := 100, coef_of_h := 1) local alpha, Ma, Mp, Mt, Mpp, Mtt, M0, M1, E, l; l := 1/(maximize(K(x, s), x = a .. b, s = a .. b)*abs(varphi(b)-varphi(a)-psi(b)+psi(a))); if l <= abs(lambda) b)*Ma+2*maximize((diff(K(x, s), x))/(diff(varphi(x), x)), x = a .. b, s = a .. b)*Mp+maximize(K(x, s), x = a .. b, s = a .. b)*Mpp; if psi(x) <> 0 then M1 := maximize((diff((diff(K(x, s), x))/(diff(psi(x), x)), x))/(diff(psi(x), x)), x = a .. b, s = a .. b)*Ma+2*maximize((diff(K(x, s), x))/(diff(psi(x), x)), x = a .. b, s = a .. b)*Mt+maximize(K(x, s), x = a .. b, s = a .. b)*Mtt else M1 := 0 end if; E := (1/12)*M0*(varphi(b)-varphi(a))*Omega(varphi, a, b, n, coef_of_h)^2+(1/12)*M1*(psi(b)-psi(a))*Omega(psi, a, b, n, coef_of_h)^2; E end proc;

This algorithm is to find error for the kernel K(x, s).

```
Difference_V := proc(K, φ, ψ, f, a, b, λ := 1, n := 100, coef_of_h := 1)
    local h := (b − a)/n, x :: list, A :: list, B :: list, Id, Temp1, Temp2, i, j, C, F, E, Pr, v :: list, V;
    for i from 0 to n do
        x_i := a + i·h ;
    end do ;
    A_0 := λ/2 ·K(x, x_0)·(φ(x_1) − φ(x_0) − ψ(x_1) + ψ(x_0));
    A_n := λ/2 ·K(x, x_n)·(φ(x_n) − φ(x_(n−1)) − ψ(x_n) + ψ(x_(n−1)));
for i from 1 to n − 1  do A_i := λ/2 · K(x, x_i)·(φ(x_(i+1)) − φ(x_(i−1)) − ψ(x_(i+1))
    + ψ(x_(i−1))); end do;
Id := IdentityMatrix( n + 1);
Temp1 := Matrix( n + 1);
for i from 0 to  n do for j from 0 to n do Temp1(i + 1, j + 1) := subs(x = x_i, A_j); end do end
```

Fig. 2.Algorithm for V(x)

IV. IMPLEMENTATION OF ERROR ANALYSIS

Example 1: Perform error analysis for the equation

$$u(x)=\frac{2}{5}\int_0^1\left(xs^2+x^2 s\right) u(s)\, d\left(\sqrt[3]{s}\right)-\frac{1}{4}\cdot x^2-\frac{1}{6}\cdot x+x\sqrt{x}$$

The equation will be solved by Maple and the results given in a table. Here $K:=(x, s)->xs^2+x^2 s:$

$f:=x->-\frac{1}{4}x^2-\frac{x}{6}+x\sqrt{x}:$, $\varphi:=x->\sqrt[3]{x}:$, $\psi:=0:$,

$a := 0.0 :, \ b := 1.0 :, \ \lambda := 0.4 :$ for $n = 4$, $\mathrm{h} = \dfrac{\mathrm{b - a}}{4} = 0.25$,

$\mathrm{x}_1 = 0, \mathrm{x}_2 = 0.25, \ \mathrm{x}_3 = 0.50, \ \mathrm{x}_4 = 0.75, \mathrm{x}_5 = 1$.

In the table 1 the approximate solutions, exact solutions and the determinant of the formed matrix are given for some values of n.

TABLE I. APPROXIMATE SOLUTIONS, ERROR FUNCTIONS AND ERRORS FOR EXAMPLE 1

n	V(x)	ū(x)	E(h)
4	$0.34953x + 0.60819x^2$	$-0.14116x - 0.21577x^2 + x\sqrt{}$	11.94348292
16	$0.13999x + 0.24385x^2$	$-0.1443x - 0.21979x^2 + x\sqrt{x}$	5.500020907
64	$0.05685x + 0.09774x^2$	$-0.1444x - 0.21999x^2 + x\sqrt{x}$	2.254847694
256	$0.02272x + 0.03896x^2$	$-0.1444x - 0.21999x^2 + x\sqrt{x}$	0.9019247642
512	$0.01433x + 0.02456x^2$	$-0.1444x - 0.21999x^2 + x\sqrt{x}$	0.5689194703
1024	$0.00903x + 0.01548x^2$	$-0.1444x - 0.21999x^2 + x\sqrt{x}$	0.3586305578

In the table 1, the approximate solutions, error functions, and errors are given for values of $n = 4$, $n = 16$, $n = 64$, $n = 256$, $n = 512$, $n = 1024$. It can be easily seen from the table that when n increases how the error is decreases. It indicates that very close solution is found to the exact solution.

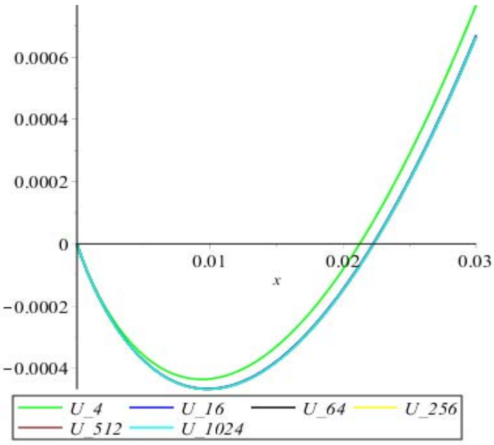

Fig. 5. Graph of approximate solutions on [0, 0.03] for example 1

In figure 5, the approximation solutions of the equation is given on [0, 0.03]. As it seen through the graph, when n increases the graphs are very close to each other. This gives a very serious hint about exact solution.

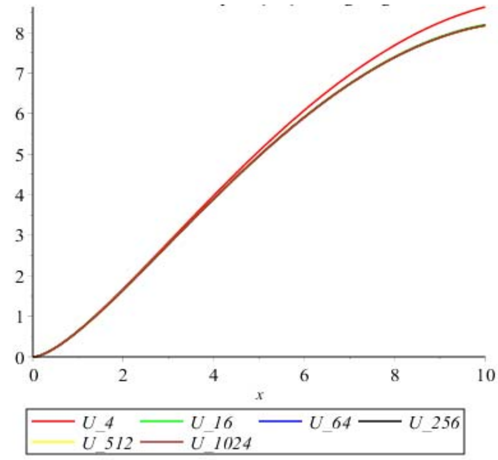

Fig. 6. Graph of approximate solutions on [0, 10] for example 1

In the figure 6, the approximation solutions of example 1 is given on [0, 10]. The behaviors of the graphs of approximate functions can be more accurate.

Example 2: Make error analyze of the equation

$$u(x) = \frac{1}{10}\int_0^1 \left(xs^2 - x^2 s\right) u(s)\, d\left(\sqrt{s} - \sqrt[3]{s}\right) - \frac{1}{4}x^2 - \frac{1}{6}x + x\sqrt{x}$$

using generalized trapezium method.

Here $K := (x, s) -> 1 + x^2 s :\ f := x -> -\dfrac{1}{4}x^2 - \dfrac{x}{6} + x\sqrt{x} :,$

$\varphi := x -> \ln\left(1 + \sqrt{x}\right) :, \ \psi := 0 :, \ a := 0.0 :, \ b := 1.0 :, \ \lambda := 0.1 :$

for $n = 4$, $\mathrm{h} = \dfrac{\mathrm{b - a}}{4} = 0.25$, $\mathrm{x}_1 = 0, \mathrm{x}_2 = 0.25, \ \mathrm{x}_3 = 0.50,$

$\mathrm{x}_4 = 0.75, \mathrm{x}_5 = 1$.

TABLE II. APPROXIMATE SOLUTIONS, ERROR FUNCTIONS AND ERRORS FOR EXAMPLE 2

n	V(x)	ū(x)	E(h)
4	$0.0001167798x - 0.0001404991x^2$	$-0.1643x - 0.2529 \cdot x^2 + x\sqrt{}$	0.2169029919
16	$0.0000520625x - 0.0000728210x^2$	$-0.1644x - 0.2529 \cdot x^2 + x\sqrt{}$	0.09144898881
64	$0.00001990x - 0.00002880x^2$	$-0.1644x - 0.2529 \cdot x^2 + x\sqrt{}$	0.03486475326
256	$0.0000757084x - 0.0000110287x^2$	$-0.1644x - 0.2529 \cdot x^2 + x\sqrt{}$	0.01325830180
512	$0.0000046879x - 0.0000068347x^2$	$-0.1644x - 0.2529 \cdot x^2 + x\sqrt{}$	0.008209591436
1024	$0.0000029111x - 0.0000042458x^2$	$-0.1644x - 0.2529 \cdot x^2 + x\sqrt{}$	0.005098129510

In the table 2, the approximate solutions, error functions, and errors are given for values of $n = 4$, $n = 16$, $n = 64$,

$n = 256$, $n = 512$, $n = 1024$. It can be easily seen from the table that when n increases how the error is decreases. It indicates that very close solution is found to the exact solution.

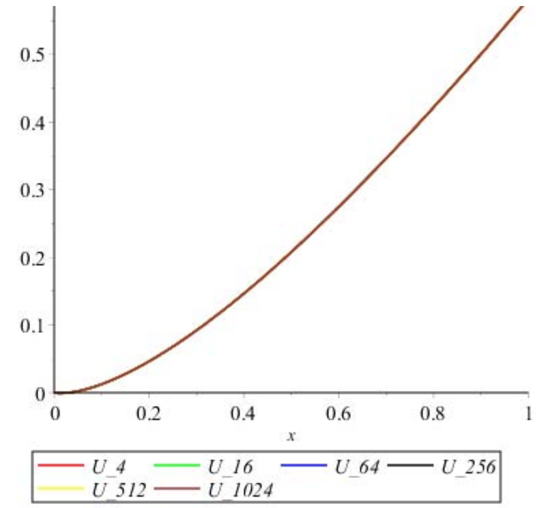

Fig. 7. Graph of approximate solutions on [0, 1] for example 2

In figure 7, the approximation solutions of the equation is given on [0, 1]. As it seen through the graph, when n increases the graphs are very close to each other. This gives a very serious hint about exact solution.

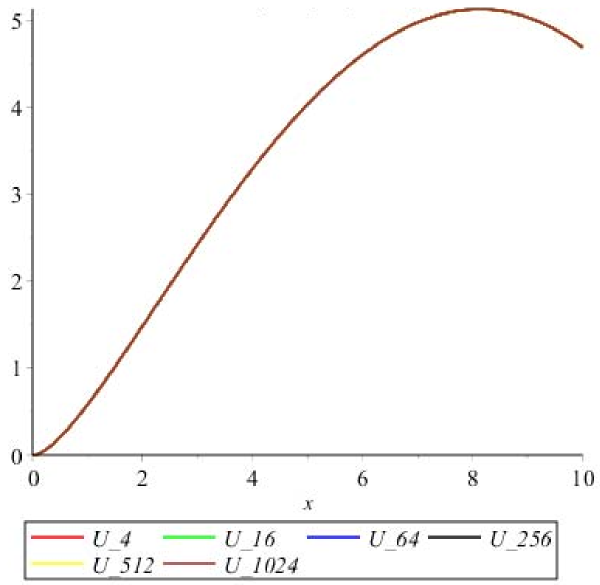

Fig. 8. Graph of approximate solutions on [0, 10] for example 2

In the figure 8, the approximation solutions of example 2 is given on [0, 10]. The behaviors of the graphs of approximate functions can be more accurate.

V. CONCLUSION

In this work, a computation algorithm has been proposed for error analysis of the linear Fredholm-Stiltjes integral equations solved by trapezium approximation method [20]. The method has been reinforced by numerical examples. Maple software has been used for developing algorithm and computations. Some basic examples have been chosen to see accurate results. Very close results to exact solutions have been found. The method has been applied only to linear equations. It can be applied to other kind of Stieltjes integral equations.

REFERENCES

[1] Brauer F, On a nonlinear integral equation for population growth problems, *SIAM J. Math. Anal.*, 6 **(1975)**: 312-317.

[2] Brauer F, Constant rate harvesting of populations governed by Volterra integral equations, *J. Math. Anal. Appl.*, 56 **(1976)**: 18-27

[3] Büyükaksoy A and Alkumru A, Multiple diffraction of plane waves by a soft/hard strip, Journal of Engineering Mathematics, 29(2) **(1995)**: 105- 120.

[4] Dehghan M and Shakeri F, Application of He's variational iteration method for solving the Cauchy reaction-diffusion problem, J. Comput. Applied Math., 214 **(2008)**: 435-446.

[5] Muslu I, Minimum Enerji ile Lineer Sistemde Yönetim Metodu, Yüksek Lisans Tezi, KTMÜ, Bişkek, 2010.

[6] Bloom F, Asymptotic bounds for solutions to a system of damped integro-differential equations of electromagnetic theory, Journal of Mathematical Analysis and Applications, 73(2) **(1980)**: 524-542K. Stroyan, Interactive Multivariable Calculus, http://itsnt711.iowa.uiowa.edu:8080/webMathematica/iMultiCal

[7] Kopeikin ID and Shishkin VP, Integral form of the general solution of equations of steady-state thermoelasticity. Journal of Applied Mathematics and Mechanics (PMM U.S.S.R.), 48(1) **(1984)**: 117-119.

[8] Aleksandrov VM and Kovalenko EV, Mathematical method s in the displacement problem," *Inzhenernyj Zhurnal. Mekhanika Tverdogo Tela*, 2 **(1984)**: 77–89.

[9] Agarwal RP, O'Regan D and Wong PJY, Positive Solutions of Differential, Difference and Integral Equations, KAP, Dordrecht, 1999.

[10] Arikogulu A and Ozkol I, Solutions of boundary Value Problems for Integro-Differential Equations by Using Di_erential Transform Method. Appl. Math.Comput 168 **(2005)**:1145-1158 http://www.cargo.wlu.ca/e-ECCAD2004/Maplets.pdf

[11] Wazwaz A, Linear and Nonlinear Integral Equations, Methods an Application, Higher Education Press, Berlin Heidelberg, 2011

[12] Abbasbandy S, Application of He's homotopy perturbation method to functional integral equations, Chaos, Solitons and Fractals, 31 **(2007)**: 1243-1247

[13] Atkinson K, The numerical solution of integral equations of the second kind, Cambridge University press, New York, 1997.

[14] Mustafa G and Yalcin O, On Numerical Solution of Linear Fredholm-Volterra Integro-Differential Difference Equations with Piecewise Intervals, Application and Applied Mathematics: An International Journal (AAM), 7 **(2012)**: 556-570

[15] Banaś J, Sadarangani K, Solvability of Volterra-Stieltjes operator-integral equations and their applications, Computers Math. Applic, 41 (12) **(2001)**: 1535–1544

[16] Lovelady DL, Pertubations of solutions of Stieltjes Integral Equations, Transection of the AMS.155 (1) 1971

[17] Tvrdy M, Fredholm-Stieltjes integral equations with linear constraints: duality theory and Green's function, Časopis pro pěstování matematiky, 104(4) **(1979)**: 357—369

[18] Basin VV and Serejnikov TI, Ob odnom Algoritme Reseniya Uravneniya Fredgolma- Stiltesa, Izvestiya Visih Ucebnih Zavedeniy, Matematika, 4 **(2001)**: 467-475

[19] Muslu I, Asanov A., Yanik S., Approximation Solution of Linear Fredholm-Stieltjes Integral Equations of Second Kind Using Generalized Simpson's Quadrature Method, American Research Journal of Mathematics, 1(2) **(2015)**: 1-6

[20] Muslu I, Asanov A., Approximation Solution of Linear Fredholm-Stieltjes Integral Equations of Second Kind Using Generalized Trapezoid Quadrature Approximation Method", Исследования по интегро-дифференциалным уравнениям, 46, pp.195-199, 201